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13. ABSTRACT

In a recent paper L. Lovász has settled in the affirmative a conjecture due to Berge that had be . outstanding in graph theory for over a decade, the perfect graph conjecture. In an earlier paper the author had obtained a closely related result, the pluperfect graph theorem. Connections between these two theorems are discussed, and in particular, the central role played by the notion of pluperfection in studying perfection in graphs is emphasized.

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ON THE PERFECT GRAPH THEOREM

by

D. R. Fulkerson

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III

In a recent brilliant paper [7] L. Lovász has settled in the affirmative a conjecture due to Berge [1,2] that had been outstanding in graph theory for over a decade, the perfect graph conjecture. In his paper Lovász gives two separate proofs of the conjecture. The first proof uses combinatorial results I had obtained earlier about anti-blocking pairs of polyhedra [3,4,5]. Concerning this proof, Lovász states: "It should be pointed out that thus the proof consists of two steps, and the most difficult second step was done first by Fulkerson." I would be less than candid if I did not say that I agree with this remark, at least in retrospect. But the fact remains that, while part of my aim in developing the anti-blocking theory had been to settle the perfect graph conjecture, and that while I had succeeded via this theory in reducing the conjecture to a simple lemma about graphs [3,4], and had developed other seemingly more complicated equivalent versions of the conjecture [3,4,5], I eventually began to feel that the conjecture was probably false and thus spent several fruitless months trying to construct a counterexample. It is not altogether clear to me now just why I felt the conjecture was false, but I think it was due mainly to one equivalent version I had found [4,5], a version that does not explicitly mention graphs at all.

Theorem 1. *Let* A *be a* $(0,1)$-*matrix such that the linear program* $yA \geq w$, $y \geq 0$, $\min 1 \cdot y$ (*where* $1 = (1,\ldots,1)$) *always has an integer solution vector* y *whenever* w *is a* $(0,1)$-*vector. Then this program always has an integer solution vector* y *whenever* w *is a nonnegative integer vector*.

While this version of the perfect graph theorem says nothing explicitly about graphs, it should be noted that a (0,1)-matrix A which satisfies the hypothesis of Theorem 1 must be the incidence matrix of (maximal) cliques vs. vertices of a graph G, and hence the hypothesis can be rephrased to say: For any vertex-generated subgraph H of G (the vertices of H correspond to components of w that are 1), one can cover all vertices of H by assigning integer (and hence (0,1)-valued) weights (components of y) to cliques of H just as efficiently as one can by allowing fractional weights on cliques of H, efficiency being measured in terms of the component-sum of y. The conclusion then asserts that under these conditions, if one assigns vertices of G arbitrary nonnegative integer weights (components of w), the same phenomenon results: the rational covering program $yA \geq w$, $y \geq 0$, $\min 1 \cdot y$, always has an integer solution.

My proof of the equivalence of this assertion and the perfect graph conjecture was based on certain facts about anti-blocking pairs of polyhedra and on linear-programming duality arguments. This proof has not yet been published. But Lovász has given a combinatorial proof of Theorem 1 in [7], using the perfect graph theorem as one main tool.

At this point I should say what the perfect graph theorem asserts. Let G be a graph and let $\gamma(G)$, $\lambda(G)$, $\pi(G)$, and $\omega(G)$ denote respectively the chromatic number of G (the minimum number of anti-cliques (independent sets of vertices) that cover G), the clique number of G (the size of a largest clique in G), the partition number of G (the minimum number of cliques that cover G), and the anti-clique number (stability number) of G (the size of a largest anti-clique in G).

Clearly one always has the inequalities $\gamma(G) \geq \lambda(G)$, $\pi(G) \geq \omega(G)$. The graph G is γ-perfect if $\gamma(H) = \lambda(H)$ for every vertex-generated subgraph H of G; G is π-perfect if $\pi(H) = \omega(H)$ for every vertex-generated subgraph H of G; G is perfect if it is both γ-perfect and π-perfect, i.e. if both G and its complementary graph $\overline{G}$ are γ-perfect (or π-perfect).

Theorem 2 (Perfect graph theorem). *If G is γ-perfect (or π-perfect), then G is perfect.*

A stronger form, one that is still open, asserts that G is perfect if and only if neither G nor its complement $\overline{G}$ contains an "odd hole" (an odd chordless circuit of size ≥ 5). Recently A. C. Tucker has proved this for planar graphs G [9]; planarity is a severe restriction, however, in studying perfection in graphs.

There are numerous classes of graphs that are known to be perfect. Some of the better-known examples are interval graphs, rigid-circuit graphs, comparability graphs, and in particular, bipartite graphs.

Now let A and B be the m by n and r by n incidence matrices of the families of (maximal) cliques and (maximal) anti-cliques, respectively, of G, where A has rows $a^1,\ldots,a^m$, B has rows $b^1,\ldots,b^r$. Define functions $\gamma_G(w)$, $\lambda_G(w)$, $\pi_G(w)$, $\omega_G(w)$, where w is a nonnegative integer n-vector, as follows. Let $\gamma_G(w)$ be the minimum in the integer linear program $yB \geq w$, $y \geq 0$, $\min 1\cdot y$; let $\pi_G(w)$ be the minimum in the integer linear program $yA \geq w$, $y \geq 0$, $\min 1\cdot y$; let $\lambda_G(w) = \max_{1\leq i\leq m} a^i\cdot w$; let $\omega_G(w) = \max_{1\leq j\leq r} b^j\cdot w$. Again one has the inequalities

$\gamma_G(w) \geq \lambda_G(w)$, $\pi_G(w) \geq \omega_G(w)$. Say that G is γ-pluperfect if $\gamma_G(w) = \lambda_G(w)$ for all w, that G is π-pluperfect if $\pi_G(w) = \omega_G(w)$ for all w, and that G is pluperfect if it is both γ-pluperfect and π-pluperfect. (Note that γ-perfection, for example, would require only $\gamma_G(w) = \lambda_G(w)$ for all (0,1)-vectors w, rather than for all nonnegative integer vectors w.) One of the main combinatorial consequences of the theory of anti-blocking pairs of polyhedra is [3,4,5]:

Theorem 3 (Pluperfect graph theorem). *If G is γ-pluperfect (or π-pluperfect), then G is pluperfect.*

Thus I knew early in 1970 that to prove the perfect graph conjecture, it would suffice to show that if G is π-perfect, say, then G is also π-pluperfect. For this it would suffice, by an easy induction, to prove the following simple lemma [3,4].

Lemma (Replication lemma). *If G is π-perfect, and if we duplicate an arbitrary vertex in G, the resulting graph G' is π-perfect.*

Actually I knew more: Namely that the truth or falsity of the perfect graph conjecture rested entirely on the truth or falsity of the replication lemma. I tried for awhile to prove this lemma, without success, and then, as was mentioned earlier, became convinced on other grounds that the perfect graph conjecture was probably false, and began to look for a graph that was perfect but not pluperfect. (I knew that it would do no good to look at known classes of perfect graphs, since I had been able to prove that all

of these were pluperfect.) The fact is that such graphs don't exist, of course. After some months of sporadic effort along these lines, I quit working on the perfect graph conjecture, thinking that I would come back to it later. There were other aspects of anti-blocking pairs of polyhedra, and of blocking pairs of polyhedra, that I wanted to study, and, in any event, I felt that the pluperfect graph theorem was a beautiful result in its own right.

In the spring of 1971 I received a postcard from Berge, who was then visiting the University of Waterloo, saying that he had just heard that Lovász had a proof of the perfect graph conjecture. This immediately rekindled my interest, naturally, and so I sat down at my desk and thought again about the replication lemma. Some four or five hours later, I saw a simple proof of it.

Proof of replication lemma. Suppose G is π-perfect, and let G' be formed from G by duplicating vertex v, i.e. adjoin v' to G and join v' to all neighbors of v in G. It is enough to show that $\pi(G') = \omega(G')$. If v belongs to a maximum cardinality anti-clique in G, this is trivial, since $\omega(G') = \omega(G)+1$. Suppose on the other hand that v is in no maximum anti-clique of G. We show in this case that there is a minimum cover of G by cliques in which v is doubly covered. Let $C_1,\ldots,C_k$ be a minimum cover of G by cliques with $v \in C_1$. Suppress $C_1 - v$ (which is nonempty) to obtain graph H. Now $\omega(H) = k-1$, since $C_2,\ldots,C_k$ cover $H-v$ and v is in no maximum anti-clique of G. Since H is π-perfect, we can cover H by cliques $K_2,\ldots,K_k$ of H, with $v \in K_2$, say. These are cliques in G, and hence G is covered by C_1, $K_2,\ldots,K_k$, and $v \in C_1$, $v \in K_2$.

This proves the assertion, whence it follows that $\pi(G') = \omega(G')$ in this case as well, proving the lemma.

As can be seen, there is nothing deep or complicated about the proof of this lemma. Perhaps the fact that I saw a proof of it only after knowing it had to be true may say something about the psychology of invention (or, better yet, anti-invention) in mathematics, at least for me.

A few weeks later I received a pre-print of Lovász's paper. One of his two proofs of the perfect graph theorem uses the pluperfect graph theorem and a stronger version of the replication lemma: The stronger version asserts that replacing a vertex v of a π-perfect graph G by a π-perfect graph H yields a π-perfect graph G'. His second proof, which is short and elegant, also uses his replacement lemma. This proof does not explicitly invoke the pluperfect graph theorem, but it again brings out the central role played by the notion of pluperfection in studying perfection. He describes the theorem and proof in terms of hypergraphs [2] (hypergraph = family of sets), but the description in this form is only apparently more general, not actually so (just as Theorem 1 is more general in appearance only than a theorem about graphs).

Lovász's second proof, restated in purely graph-theoretic language, runs as follows. Suppose that G is π-perfect, so that $\pi(H) = \omega(H)$ for every vertex-generated subgraph of G. Again it is sufficient to show that $\gamma(G) = \lambda(G)$. The proof can be viewed as an induction on $\lambda(G) = k$. The case $k = 1$ gives no difficulty. Consider the case $k > 1$. It is enough to find a maximal anti-clique D in G such that the graph $G-D$ obtained from G by suppressing vertices of D has $\lambda(G-D) = k-1$. Assume, on the

contrary, that for every maximal anti-clique D in G, there is a clique $C(D)$, no vertex of which is in D, of size k, i.e., assume that graph $G-D$ has a clique $C(D)$ of size k. Consider the weight vector w having components $w(v)$, one for each vertex v of G, defined as follows: Let $w(v)$ be the number of times v occurs in the family of sets $C(D)$, as D ranges over all maximal anti-cliques of G. Let G' be the graph formed from G by replacing each vertex v in G by $w(v)$ replicates of v. (If $w(v) = 0$, this means that v is suppressed in forming G'.) It follows from the replication lemma (or from Lovász's replacement lemma) that G' is π-perfect. If G has m maximal anti-cliques $D_1, D_2, \ldots, D_m$, then G' has mk vertices. Since the size of a maximum clique in G' is at most k, we have $\pi(G') \geq m$. On the other hand, a given anti-clique D_i of G contains at most one vertex of $C(D_j)$, and no vertex of $C(D_i)$, and hence

$$\begin{aligned}\omega(G') &= \max_{1\leq i\leq m} \sum_{v\in D_i} w(v) \\ &= \max_{1\leq i\leq m} \sum_{j=1}^{m} |D_i \cap C(D_j)| \\ &\leq m-1,\end{aligned}$$

contradicting the fact that G' is π-perfect. This proves Theorem 2.

Observe that the contradiction obtained in the proof is gotten by a shrewd choice of the integer weight vector w.

Another characterization of pluperfect graphs (and hence of perfect graphs) that I gave in [3,4] and called the max-max inequality, can be

described as follows. Again let A and B be the m by n and r by n incidence matrices of the families of maximal cliques and anti-cliques of a graph G, where A has rows $a^1,\dots,a^m$, one for each maximal clique, and B has rows $b^1,\dots,b^r$, one for each maximal anti-clique. Let ℓ and w be nonnegative n-vectors, whose components correspond to vertices of G.

Theorem 4. The graph G is pluperfect if and only if

$$(*) \qquad \left(\max_{1\le i\le m} a^i\cdot\ell\right)\left(\max_{1\le j\le r} b^j\cdot w\right) \ge \ell\cdot w$$

for all nonnegative weight vectors ℓ and w.

In other words, the "weight" of a largest clique, computed using ℓ, times the "weight" of a largest anti-clique, computed using w, is at least equal to the inner product $\ell\cdot w$ of the weight vectors ℓ and w. (More generally, the max-max inequality, together with the assumption that $a^i\cdot b^j \le 1$ for all rows a^i of A and b^j of B, characterizes the anti-blocking relation for nonnegative matrices A and B, i.e., characterizes anti-blocking pairs of polyhedra.) In another and more recent paper [8] Lovász has proved the following simplification of Theorem 4.

Theorem 5. The graph G is perfect (pluperfect) if and only if (*) holds for all (0,1)-vectors $\ell = w$.

Thus a graph G is perfect (pluperfect) if and only if $\lambda(H)\omega(H) \ge |H|$ for all vertex-generated subgraphs H of G, where $|H|$ denotes the number of vertices of H.

In the context of anti-blocking pairs of matrices or polyhedra, Theorem 5 raises the following question. If we assume only that A is the (0,1)-incidence matrix of m pairwise noncomparable subsets of an n-set, that matrix B is a (0,1)-matrix whose rows satisfy $a^i \cdot b^j \le 1$ for all $i = 1,\ldots,m$, $j = 1,\ldots,r$, and that

$$(**) \qquad (\max a^i \cdot w)(\max b^j \cdot w) \ge w \cdot w$$

for all (0,1)-vectors w, then it can be shown, using the Gilmore characterization of clique matrices [6], that A is the clique matrix of a graph G, B is the anti-clique matrix of G, and hence by Lovász's result, G is perfect, i.e., A and B are an anti-blocking pair of matrices. It would be interesting to know under what conditions on a (0,1)-matrix A and a matrix B satisfying $a^i \cdot b^j \le 1$, the max-max inequality (*) for A and B can be replaced by (**), and still get the conclusion that A and B are an anti-blocking pair.

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